

Projective Root-Locus: An Extension of Root-Locus Plot to the Projective Plane

Francisco Mota

Departamento de Engenharia de Computação e Automação
Universidade Federal do Rio Grande do Norte – Brasil
e-mail:mota@dca.ufrn.br

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Abstract

In this paper we present an extension of the classical Root-Locus (RL) method where the points are calculated in the real projective plane instead of the conventional affine real plane; we denominate this extension of the Root-Locus as “Projective Root-Locus (PjRL)”. To plot the PjRL we use the concept of “Gnomonic Projection” in order to have a representation of the projective real plane as a semi-sphere of radius one in \mathbb{R}^3 . We will see that the PjRL reduces to the RL in the affine XY plane, but also we can plot the RL onto another affine component of the projective plane, like ZY affine plane for instance, to obtain what we denominate complementary plots of the conventional RL. We also show that with the PjRL the points at infinity of the RL can be computed as solutions of a set algebraic equations.

Index terms— Root-Locus, Projective Plane, Gnomonic Projection, Algebraic Geometry, Affine Algebraic Variety, Projective Algebraic Geometry, Ideal of Polynomials, Grobner Basis.

1 Introduction

The Root-Locus (RL) method is a classical tool that has been used extensively in the feedback control literature for studying the stability and performance of a closed loop linear feedback system. It consists of a parametric plot of the roots of the polynomial $p(s) = d(s) + kn(s)$ in the complex plane, as the parameter k spans \mathbb{R} ; d and n are fixed coprime polynomials, and d is monic with degree, in general, greater than the degree of n . In fact, the polynomial p represents the denominator of the transfer function of a closed-loop feedback system that has the (irreducible) proper rational function $G(s) = n(s)/d(s)$ as a linear time invariant plant model and k as a (proportional type) controller (see Figure 1) and that is why we use the terminology the “RL for $G(s)$ ”. To plot the RL for a given $G(s)$, most control theory textbooks presents a set of rules that allow us to make an approximate sketch of the plot ([1]), but a detailed plot, nowadays, in general, is obtained using a computer software that evaluates the roots of p , using numerical techniques, for a given range of the parameter k in \mathbb{R} (e.g. Scilab ([2])). In Figure 2 we show the plot of the RL for the plant $G(s) = (s + 1)/s^2$, for some range of $k \in \mathbb{R}$.

The motivation to use the projective plane to analyze the RL method is that the RL plot for a given $G(s)$ can have points at infinity: the parameter k itself has to reach an “infinite value” in order we can obtain the “terminal” points of the RL, that can, in turn, be finite (zeros of $G(s)$) or to be located at infinity. In this way, using the concepts of projective real line and projective plane we can account for these “infinite” points, and also obtain complementary plots of the RL where points at infinity can be plotted at a finite position onto an affine plane. We denominate this extension of the RL to the projective plane as “Projective Root-locus (PjRL)” and, in spite of its abstract definition, we will show that it can be relatively easy to

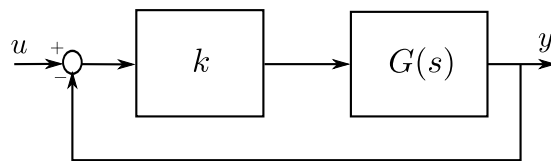


Figure 1: Control Feedback Loop with a Proportional Controller

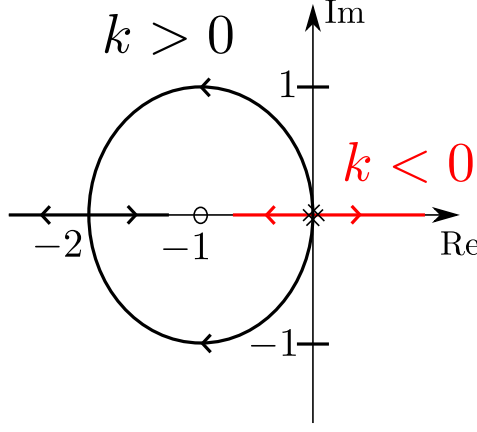


Figure 2: Root-Locus for $G(s) = (s+1)/s^2$

obtain the PjRL for $G(s)$ using a computer algebra software. Below we introduce the definitions and notation to be used along the paper:

\mathbb{R} , \mathbb{C} and $\mathbb{R}[x_1, x_2, \dots, x_n]$: Represents the field of real numbers, the field of complex numbers and the ring of polynomials with coefficient's in \mathbb{R} and with indeterminates (x_1, x_2, \dots, x_n) , respectively.

Projective (real) line: The projective line $\mathbb{P}^1(\mathbb{R})$ is the set of “slopes” y/x , $(x, y) \neq (0, 0) \in \mathbb{R}^2$ and $1/0 = \infty$. So, if $k \in \mathbb{P}^1(\mathbb{R})$, then $k = k_n/k_d$, and $k = \infty$ corresponds to $k_n = 1$ and $k_d = 0$ ($k_n = k_d = 0$ is not allowed). We note that $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

Projective (real) plane: The projective plane $\mathbb{P}^2(\mathbb{R})$ is the set of equivalence classes of all *nonzero* triples $(x, y, z) \in \mathbb{R}^3$ under the equivalence relation: $(\alpha_1, \alpha_2, \alpha_3) \equiv (\beta_1, \beta_2, \beta_3)$ if $\alpha_i = \lambda\beta_i$, for some $\lambda \neq 0$. We represent the equivalence class of (x, y, z) by $(x : y : z)$, that is denominated “homogeneous coordinate” of (x, y, z) . We note that $\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup H$, where H represents the plane at infinity and it is disjoint from \mathbb{R}^2 . Mathematically, $\mathbb{R}^2 = \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R})\}$, the XY plane, and $H = \{(x : y : 0) \in \mathbb{P}^2(\mathbb{R})\}$. In fact, H has two “types” of points, namely, $(1 : m : 0)$ and $(0 : 1 : 0)$, where $(1 : m : 0)$ represents the point of intersection of all (XY) lines with finite slope “ m ” and $(0 : 1 : 0)$ represents the intersection of all lines with infinite slope (vertical lines).

Homogeneous polynomial A polynomial (in several variables) is homogeneous when all of its nonzero terms (monomials) have the same total degree. One important fact about a homogeneous polynomial $p \in \mathbb{R}[x_1, x_2, \dots, x_n]$ is that $p(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^d p(x_1, x_2, \dots, x_n)$, where d is the total degree of p . We always can turn a non-homogeneous polynomial (q) into a homogeneous one (q^h) by adding a new variable (x_{n+1}), with the following procedure: $q^h(x_1, \dots, x_n, x_{n+1}) = x_{n+1}^d q(x_1/x_{n+1}, x_2/x_{n+1}, \dots, x_n/x_{n+1})$, where d is the total degree of q ; this process is denominated “homogenization” of q . We can always “de-homogenize” q^h by setting $x_{n+1} = 1$ and recover back q .

Affine Algebraic Variety An affine (real) algebraic variety \mathcal{V} generated by a set of m polynomials, $p_i \in \mathbb{R}[x_1, x_2, \dots, x_n]$, is a subset of the affine plane \mathbb{R}^n composed by the coordinates (x_1, x_2, \dots, x_n) that are simultaneous *real* roots of the m generating polynomials, that is $p_i(x_1, x_2, \dots, x_n) = 0$, $i = 1, \dots, m$.

Projective Algebraic Variety A projective (real) algebraic variety \mathcal{W} generated by a set of m *homogeneous* polynomials, $p_i \in \mathbb{R}[x_1, x_2, \dots, x_n, x_{n+1}]$, is a subset of the projective plane $\mathbb{P}^n(\mathbb{R})$ composed by the *homogeneous* coordinates $(x_1 : x_2 : \dots : x_n : x_{n+1})$ such that $(x_1, x_2, \dots, x_n, x_{n+1})$ is a simultaneous *real* root of the m generating homogeneous polynomials, that is $p_i(x_1, x_2, \dots, x_n, x_{n+1}) = 0$, $i = 1, \dots, m$. We note that if $(x'_1, \dots, x'_n, x'_{n+1})$ is any member of the equivalence class $(x_1 : x_2 : \dots : x_n : x_{n+1})$, it is also a simultaneous root of p_i , $i = 1, \dots, m$, since p_i is homogeneous of degree d : $x'_j = \lambda x_j$, then $p_i(x'_1, \dots, x'_n, x'_{n+1}) = \lambda^d p_i(x_1, x_2, \dots, x_n, x_{n+1}) = 0$.

Ideal of Polynomials: Let be $\{p_1, p_2, \dots, p_t\}$ a set of polynomials in $\mathbb{R}[x_1, x_2, \dots, x_n]$. The set of polynomials $I \subseteq \mathbb{R}[x_1, x_2, \dots, x_n]$ defined by

$$I = \sum_{i=1}^t h_i p_i, \quad h_i \in \mathbb{R}[x_1, x_2, \dots, x_n]$$

is an ideal of $\mathbb{R}[x_1, x_2, \dots, x_n]$, and $\{p_1, p_2, \dots, p_t\}$ is denominated a *generating set* for I ; in this case we write $I = \langle p_1, p_2, \dots, p_t \rangle$. A Grobner Basis for the ideal I is a particular kind of generating set that allows many important properties of the ideal to be deduced easily. Given a generating set $\{p_1, p_2, \dots, p_t\}$ for I , we can obtain a Grobner basis $\{g_1, g_2, \dots, g_s\}$ for I algorithmically (see [3, Ch. 2]).

For more details about the concepts above see ([3], [4]).

2 The Projective Root-Locus - PjRL

As discussed in Introduction, the conventional RL for an irreducible proper rational function $G(s) = n(s)/d(s)$ is a plot of the roots of the polynomial $p(s) = d(s) + kn(s)$, when $k \in \mathbb{R}$; that is, we solve

$$d(s) + kn(s) = 0 \quad (1)$$

for each $k \in \mathbb{R}$ and plot its roots in the affine plane \mathbb{R}^2 . But, since the parameter k belongs to \mathbb{R} , to analyze the situation where $k \rightarrow \pm\infty$, we will modify Equation (1) slightly by considering $k \in \mathbb{P}^1(\mathbb{R})$. So, following the definition of $\mathbb{P}^1(\mathbb{R})$, we set $k = k_n/k_d$ in (1) and clear the denominator to obtain:

$$k_d d(s) + k_n n(s) = 0. \quad (2)$$

We note that setting $k_d = 1$ in Equation (2) we recover Equation (1) and setting $k_d = 0$, that is $k = \infty$, corresponds to $n(s) = 0$ in Equation (2), or the finite “terminal” points of the RL (zeros of $G(s)$). We then see that the effect of passing from $k \in \mathbb{R}$ to $k \in \mathbb{P}^1(\mathbb{R})$ is just that of including the roots of $n(s)$, the finite terminal points, into the RL. As we will see in the next sections, the “infinite” terminal points of the RL will only appear when we extrapolate from \mathbb{R}^2 to $\mathbb{P}^2(\mathbb{R})$. We also note that we can treat the case where the degree of d is less than the degree of n in the same fashion we treat the case where the degree of d is greater than the degree of n by just exchanging the positions of k_d and k_n in Equation 2. The case where the degree of d is equal the degree of n also can be treated by our approach, as shown in examples of Section 3.

Since Equation (2) may admit complex solutions, if we write $s = x + iy$ we have:

$$d(x + iy) = q_d(x, y) + ir_d(x, y) \quad \text{and} \quad n(x + iy) = q_n(x, y) + ir_n(x, y) \quad (3)$$

where q_d, r_d, q_n and r_n are polynomials in $\mathbb{R}[x, y]$. So we may rewrite (2) as:

$$[k_d q_d(x, y) + k_n q_n(x, y)] + i [k_d r_d(x, y) + k_n r_n(x, y)] = 0$$

and finding a complex solution for (2), for given pair (k_d, k_n) , is equivalent of finding a solution in \mathbb{R}^2 for the system of polynomial equations:

$$k_d q_d(x, y) + k_n q_n(x, y) = 0 \quad (4)$$

$$k_d r_d(x, y) + k_n r_n(x, y) = 0 \quad (5)$$

for each $k_n/k_d \in \mathbb{P}^1(\mathbb{R})$. It is important to stress the fact that any solution for the system (4–5) must be invariant when we pass from pair (k_n, k_d) to $(\lambda k_n, \lambda k_d)$, $\lambda \neq 0$, since they represent the same point in $\mathbb{P}^1(\mathbb{R})$. This, in fact, is true because it is equivalent to multiply Equations (4) and (5) by $\lambda \neq 0$.

To obtain the Projective Root-Locus (PjRL) we need to extend the solutions of Equations (4–5), defined above, from the affine plane \mathbb{R}^2 , to the projective plane $\mathbb{P}^2(\mathbb{R})$. To achieve this goal, we first need to interpret the solutions of Equation (4–5) as a real algebraic variety \mathcal{V} generated by the set of two polynomials q and r defined as:

$$q(x, y, k_d, k_n) = k_d q_d(x, y) + k_n q_n(x, y) \quad (6)$$

$$r(x, y, k_d, k_n) = k_d r_d(x, y) + k_n r_n(x, y). \quad (7)$$

Since the polynomials q and r are defined in $\mathbb{R}[x, y, k_d, k_n]$ we would have a variety in \mathbb{R}^4 , i.e. $\mathcal{V} \subset \mathbb{R}^4$; but, since k_n/k_d is defined in $\mathbb{P}^1(\mathbb{R})$, in fact, we have $\mathcal{V} \subset \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$. Based on this, we could abstractly interpret the RL as the projection (represented by \mathcal{V}_k) of \mathcal{V} onto \mathbb{R}^2 , since each point of the RL is a solution of (4–5) for a fixed $k = k_n/k_d \in \mathbb{P}^1(\mathbb{R})$. We note that, for each $k \in \mathbb{P}^1(\mathbb{R})$, \mathcal{V}_k is an (finite) affine real variety defined in \mathbb{R}^2 , by the solutions of Equations (4–5), or equivalently, by the roots of Equation (2).

Now we proceed with the question of extrapolating the RL from the affine plane (\mathbb{R}^2) to the projective plane ($\mathbb{P}^2(\mathbb{R})$). Our approach will follow the two steps below:

- (1) Extrapolate the algebraic variety $\mathcal{V} \subset \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$, defined above, to obtain a projective algebraic variety $\mathcal{W} \subset \mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$;

- (2) Obtain the projection of \mathcal{W} onto $\mathbb{P}^2(\mathbb{R})$. This projection, represented by \mathcal{W}_k , $k \in \mathbb{P}^1(\mathbb{R})$, it will be what we denominate PjRL.

To obtain \mathcal{W} from \mathcal{V} , we could simply homogenize the polynomials q and r , presented in Equations (6–7), and obtain a projective variety \mathcal{W} in $\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$, now generated by the homogenized polynomials $q^h(x, y, z, k_d, k_n) = z^d q(x/z, y/z, k_d/z, k_n/z)$ and $r^h(x, y, z, k_d, k_n) = z^e r(x/z, y/z, k_d/z, k_n/z)$, as defined in Introduction. The projective variety obtained this way reduces to \mathcal{V} in $\mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$, since the process of de-homogenization of q^h and r^h will restore back the polynomials q and r . The flaw with this approach is that the process of simply homogenizing the generating polynomials for \mathcal{V} , in general, creates a projective variety that is “greater” than the necessary, in the sense that it may add points at infinity to the original variety, other than the existing ones (see [3, Ch. 8]). Then, in fact, \mathcal{W} must be the “projective closure” of \mathcal{V} , that is, a *minimal* projective variety in $\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ that reduces to \mathcal{V} in $\mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$. To compute the closure of \mathcal{V} , instead of directly homogenizing the polynomials q and r that generates \mathcal{V} , we need first to compute a Grobner basis, with respect a *graded* monomial order, for the ideal $I = \langle q, r \rangle$ (see [3, Ch. 8]). The projective closure of \mathcal{V} will be the projective variety \mathcal{W} generated by the homogenized polynomials of the obtained Grobner basis. For the sake of completeness we present the following definition for the PjRL:

Definition 2.1. (PjRL) Let be an irreducible rational function $G(s) = n(s)/d(s)$ and consider the polynomials q and r as defined in Equations (6–7). We call the PjRL of $G(s)$ the projection onto $\mathbb{P}^2(\mathbb{R})$ of the projective algebraic variety $\mathcal{W} \subset \mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$, where \mathcal{W} is generated by the set of homogenized polynomials of the Grobner basis $\{g_1, g_2, \dots, g_s\}$, with respect a graded monomial order, for the ideal $\langle q, r \rangle$. We will denote this projection by \mathcal{W}_k , where $k \in \mathbb{P}^1(\mathbb{R})$. \square

Remark 2.1. We will denote the set homogenized polynomials of the Grobner basis for $\langle q, r \rangle$ by $\{g_1^h, \dots, g_s^h\}$, where $g_i^h \in \mathbb{R}[x, y, z, k_d, k_n]$, and the variable z comes from the homogenization process, as defined in Introduction. Since we analyze \mathcal{W}_k in $\mathbb{P}^2(\mathbb{R})$, we consider $k = k_n/k_d \in \mathbb{P}^1(\mathbb{R})$ as a parameter, and the homogeneous polynomials g_i^h can be seen as defined in $\mathbb{R}[x, y, z]$. Based on the fact that $k \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, we define:

Initial points of the PjRL (\mathcal{W}_0): $k = 0/1 = 0$; that is, \mathcal{W}_0 is generated by the polynomials $\{g_1^h, g_2^h, \dots, g_s^h\}$ setting $k_n = 0$ and $k_d = 1$.

Terminal points of the PjRL (\mathcal{W}_∞): $k = 1/0 = \infty$; that is, \mathcal{W}_∞ is generated by the polynomials $\{g_1^h, g_2^h, \dots, g_s^h\}$ setting $k_n = 1$ and $k_d = 0$.

Intermediary points of the PjRL (\mathcal{W}_λ): $k = \lambda/1, \lambda \neq 0$; that is \mathcal{W}_λ is generated by the polynomials $\{g_1^h, g_2^h, \dots, g_s^h\}$ setting $k_n = \lambda \in \mathbb{R} \setminus \{0\}$ and $k_d = 1$. \square

We have the following comments regarding the results presented above:

- Calculating the Grobner basis for the ideal $\langle q, r \rangle$ is a relatively easy procedure using an algebra software available such as Macaulay2 ([6]), since we have only two polynomials that depends on four indeterminates, namely x, y, k_d and k_n .
- In the classical RL method there is a procedure for calculating the asymptotes based on the difference between the number of poles and zeros of $G(s)$. In our case, the direction of these asymptotes will appear as the solution of the algebraic equations that defines \mathcal{W}_k and it will represent points at infinity.

2.1 PjRL plot in Projective Real Plane

In order to plot the PjRL we can use the concept of gnomonic projection ([5]) to obtain a geometric representation the projective real plane. In this representation, $\mathbb{P}^2(\mathbb{R})$ is identified with a semi-sphere of radius one in \mathbb{R}^3 , as shown in Figure 3. We note that the points P (on the plane) and P' (on the sphere surface) in Figure 3 have the same homogeneous coordinates, since they belong to the same line in \mathbb{R}^3 . The points at infinity in $\mathbb{P}^2(\mathbb{R})$ are identified with the equatorial great circle, remembering that antipodal points (opposite relative to the sphere center) have the same homogeneous coordinates. Also, we note that any line in the plane $z = 1$ corresponds to a great semicircle on the semi-sphere and the the left (right) z -semi-plane corresponds to the left (right) half of the semi-sphere. So, the PjRL plot is made onto this semi-sphere, and its gnomonic projection onto the plane $z = 1$ coincides with the conventional RL (see examples in Section 3).

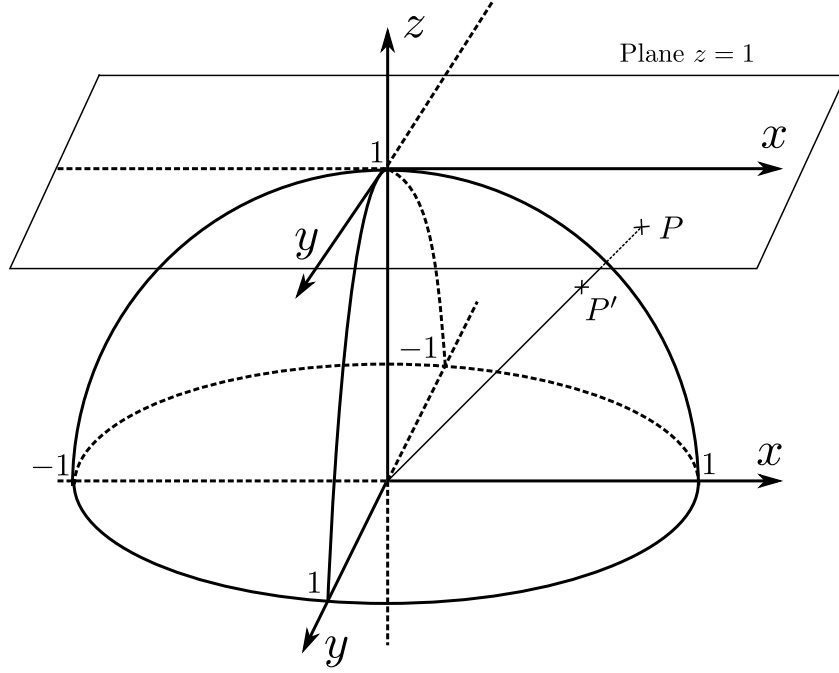


Figure 3: Gnomonic Projection of half unit sphere onto plane $z = 1$

2.2 Complementary Root-Locus plot in ZY affine plane

The equations for the PjRL will reduce to the equations for the RL when we set $z = 1$ in the set of homogeneous polynomials $\{g_1^h, \dots, g_s^h\}$ that defines \mathcal{W} . This means that when this set of polynomials is de-homogenized with respect the variable z we obtain the the RL, that is the intersection of the PjRL with the affine XY plane. But since the projective plane contains three sets that are copies of the affine planes XY , ZY and XZ , the PjRL also can give another view of the RL plot, when we analyze the intersection of it with the affine plane ZY , for instance. In this situation we de-homogenize the set of polynomials $\{g_1^h, \dots, g_s^h\}$ with respect to the variable x , instead of z , and obtain a set of polynomials that defines a new affine variety in ZY plane that we will denominate it “complementary RL”. So, the complementary RL can be seen as a gnomonic projection onto the plane $x = 1$, instead of onto the plane $z = 1$ as shown in Figure 3. Geometrically, the switch of the role of variables x and z in the complementary RL have the effect of “moving” all the points over the line $x = 0$ (in XY plane) to the infinite and “bringing” the points at infinity ($z = 0$) to a finite position. Intuitively we could state that the conventional RL is a plot as seen from the beginning ($k = 0$) while the complementary RL is a plot as seen from the end ($k = \infty$). Also there exists an interesting relation between Y crossing points in the conventional RL and asymptotes in complementary RL. More specifically, if the RL crosses the Y axis at a point, say, $(0 : y : 1)$ in XY plane for a given value of k , when we translate this point to the ZY plane it will become $(1 : y : 0)$, that is a point a infinity, in fact an asymptote with slope $y/1 = y$ in ZY plane. So, we conclude that the Y axis crossing points by the RL will become asymptotes in complementary RL, and the absolute value of the variable z will explode to infinity for the corresponding value of k . We also can make a similar analysis, now considering the intersection of the PjRL with the affine plane XZ . In examples presented in Section 3 we explore the concept of complementary RL with concrete computations.

3 Examples

As a matter of fixing ideas, we present a series of examples below.

Example 3.1. Let be $G(s) = s/(s^2 + 1)$. In this case, using notation introduced in Equation (3), we easily obtain:

$$q_d = x^2 - y^2 + 1, \quad r_d = 2xy, \quad q_n = x \quad r_n = y$$

and using the definition of q and r in (6–7), we have:

$$q(x, y, k_d, k_n) = k_d(x^2 - y^2 + 1) + k_n x, \quad r(x, y, k_d, k_n) = k_d(2xy) + k_n y \quad (8)$$

Now we compute the Grobner basis for the ideal $\langle q, r \rangle$ using the graded reversed lexicographic order ([3, pp. 56]), with $x > y > k_d > k_n$. We used the software Macaulay2 ([6]) to make the computations and obtained the Grobner basis $\{g_1, g_2, g_3, g_4\}$, where:

$$g_1(x, y, k_d, k_n) = 2xyk_d + yk_n \quad (= r) \quad (9)$$

$$g_2(x, y, k_d, k_n) = x^2k_d - y^2k_d + xk_n + k_d \quad (= q) \quad (10)$$

$$g_3(x, y, k_d, k_n) = x^2yk_n + y^3k_n - yk_n \quad (11)$$

$$g_4(x, y, k_d, k_n) = 2y^3k_d - xyk_n - 2yk_d \quad (12)$$

and the homogenized polynomials g_i^h of the Grobner basis are obtained using the procedure indicated in the Introduction:¹

$$g_1^h = z^3g_1(x/z, y/z, k_d/z, k_n/z) = 2xyk_d + yzk_n \quad (13)$$

$$g_2^h = z^3g_2(x/z, y/z, k_d/z, k_n/z) = x^2k_d - y^2k_d + xzk_n + z^2k_d \quad (14)$$

$$g_3^h = z^4g_3(x/z, y/z, k_d/z, k_n/z) = x^2yk_n + y^3k_n - yz^2k_n \quad (15)$$

$$g_4^h = z^4g_4(x/z, y/z, k_d/z, k_n/z) = 2y^3k_d - xyzk_n - 2yz^2k_d \quad (16)$$

The PjRL is the projection onto $\mathbb{P}^2(\mathbb{R})$ of the projective algebraic variety \mathcal{W} defined by the four polynomials g_i^h presented in Eqs. (13–16) above. We will represent this projection by \mathcal{W}_k , where $k \in \mathbb{P}^1(\mathbb{R})$.

- Initial points of the PjRL (\mathcal{W}_0): Setting $k_n = 0$ and $k_d = 1$ in Eqs. (13–16), we obtain: $g_1^h = 2xy$, $g_2^h = x^2 - y^2 + z^2$, $g_3^h = 0$ and $g_4^h = 2y(y^2 - z^2)$, and we see that $g_1^h = 0$ implies $x = 0$ or $y = 0$. If $x = 0$, by $g_2^h = 0$ we obtain $y^2 = z^2$. In this case we cannot have $y = 0$ or $z = 0$, since $(0, 0, 0)$ is not valid as a solution. So we can set $z = 1$ (XY plane) and obtain $y = \pm 1$. Then the initial points onto plane $z = 1$ are

$$\mathcal{W}_0 = \{(0 : 1 : 1), (0 : -1 : 1)\} = \{(0, 1), (0, -1)\}.$$

Or, onto the semi-sphere of radius one:

$$\mathcal{W}_0 = \left\{ \left(0, 1/\sqrt{2}, 1/\sqrt{2} \right), \left(0, -1/\sqrt{2}, 1/\sqrt{2} \right) \right\}.$$

- Terminal points of the PjRL (\mathcal{W}_∞): Setting $k_n = 1$ and $k_d = 0$ in Eqs. (13–16), we obtain: $g_1^h = yz$, $g_2^h = xz$, $g_3^h = y(x^2 + y^2 - z^2)$ and $g_4^h = -xyz$. To solve $g_i^h = 0$, we can simplify the set of equations calculating a Grobner basis with this set of polynomials. The new Grobner basis has three polynomials: $g_1^h = yz$, $g_2^h = xz$, $g_3^h = y(x^2 + y^2)$. Then we have two kinds of points:
 1. Points at finite position ($z \neq 0$): In this case, solving $g_i^h = 0$, we get $y = 0$ and $x = 0$. So, the homogeneous coordinate of the point is $(0 : 0 : z) = (0 : 0 : 1)$ which corresponds to $(0, 0)$ onto affine plane $z = 1$.
 2. Points at the infinite plane H ($z = 0$): In this case we obtain: $g_1^h = 0$, $g_2^h = 0$, and $g_3^h = y(x^2 + y^2)$. We must have $y = 0$ and the unique possible nonzero solution is $(x, 0, 0)$, $x \neq 0$ whose homogeneous coordinate is $(x : 0 : 0) = (1 : 0 : 0)$, which corresponds to the intersection point of the *horizontal* lines in the plane XY or a pair of points $(\pm 1, 0, 0)$ onto the equatorial great circle over the half sphere of radius one. We then have:

$$\mathcal{W}_\infty = \{(0 : 0 : 1), (1 : 0 : 0)\}.$$

- Intermediary points of the PjRL (\mathcal{W}_λ): In this case we set $k_d = 1$ and $k_n = \lambda$ in the polynomials shown in Eqs. (13–16) and recalculate the Grobner basis for the resulting set of polynomials to obtain:

$$g_1^h = y(2x + z\lambda), \quad g_2^h = x^2 - y^2 + z^2 + xz\lambda, \quad g_3^h = y(x^2 + y^2 - z^2) \quad (17)$$

We see that we must have $z \neq 0$ in (17), since $z = 0$ will imply $x = y = 0$ which is not valid as a solution; so all intermediary points are at finite positions.

¹in fact, since the polynomials g_i are already homogeneous relative to k_d and k_n , we can homogenize them relative only to x and y , and the resulting g_i^h will be the same. For example, g_1 could be homogenized as $g_1^h = z^2g_1(x/z, y/z, k_d, k_n)$.

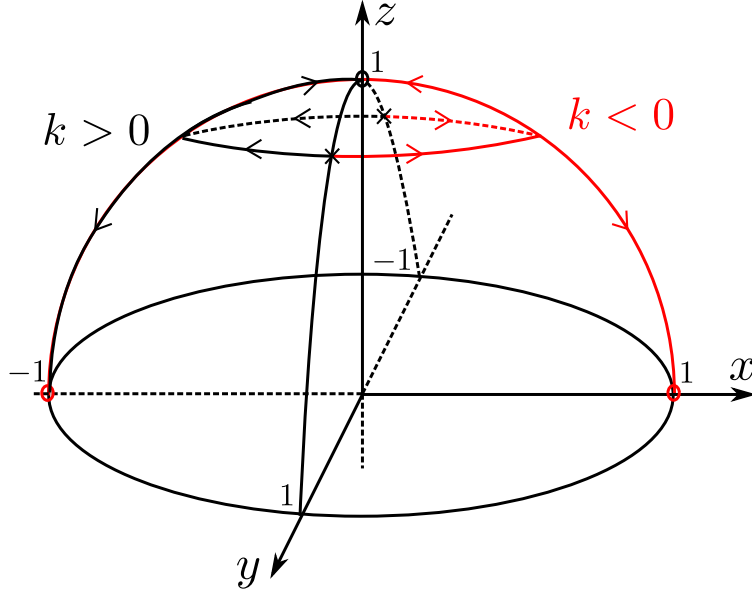


Figure 4: PjRL plot for $G(s) = s/(s^2 + 1)$

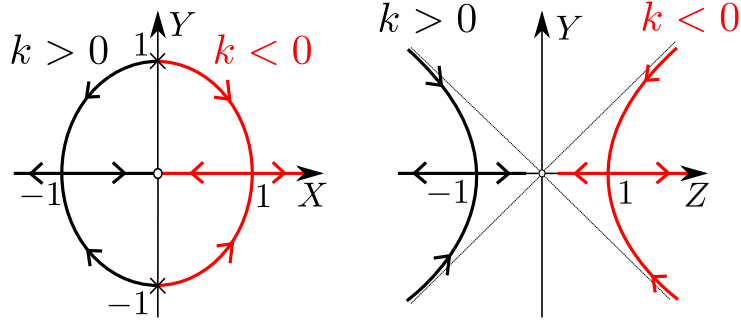


Figure 5: Conventional and complementary RL for $G(s) = s/(s^2 + 1)$

To plot the PjRL over the unit semi-sphere with radius one we need to add the equation $x^2 + y^2 + z^2 = 1$, with $z \geq 0$ to the set of equations (17). The sketch of the PjRL plot is shown in Figure 4. To obtain the conventional RL plot, we set $z = 1$ in (17) and get $g_1 = y(2x + \lambda)$, $g_2 = x^2 - y^2 + x\lambda + 1$, and $g_3 = y(x^2 + y^2 - 1)$. We note that that if $y \neq 0$, $g_3 = 0$ will require $x^2 + y^2 = 1$. The complete plot for $g_i = 0$ is shown in Figure 5.

Now we will analyze the complementary RL (in plane ZY): Switching the roles of the x and z axis in the projective plane, the coordinate $(x : y : z)$ will become $(z : y : x)$. Then, re-analyzing the initial and terminal points calculated above, we have:

$$\mathcal{W}'_0 = \{(1 : 1 : 0), (1, -1 : 0)\} \quad \text{and} \quad \mathcal{W}'_\infty = \{(1 : 0 : 0), (0 : 0 : 1)\}$$

We note that the initial points now are at infinity, that is they are asymptotes with rates ± 1 . Regarding the terminal points, there is one at infinity, that is $(1 : 0 : 0)$, or a horizontal asymptote; and other at origin $(0 : 0 : 1)$. To obtain the intermediary points, we set $x = 1$ in the polynomials shown in (17) above and we easily see that, if $y \neq 0$, we have the hyperbola $z^2 - y^2 = 1$. The plot for both conventional and complementary RL are shown in Figure 5.

Remark 3.2. We note, in this example, that if we directly homogenize the polynomials q and r , defined in Equation (8) (instead of the Grobner basis polynomials shown in Equations (9–12)), we are only left with two equations, namely:

$$q^h = x^2 k_d - y^2 k_d + x z k_n + z^2 k_d \quad \text{and} \quad r^h = 2xy k_d + y z k_n,$$

as opposed to the four equations (13–16). So, if we evaluate the terminal points \mathcal{W}_∞ using only q^h and r^h above, we would have, after setting $k_d = 0$ and $k_n = 1$, $q^h = xz$ and $r^h = yz$. It is easy to see, by these two last equations, that $(1 : 1 : 0)$ would be a possible terminal point at infinity, and this point, as we know, is

spurious. This confirms the point discussed above, that we need to evaluate the Grobner basis for the ideal $\langle q, r \rangle$.

Example 3.3. Let be $G(s) = (s + 1)/s^2$, so we have

$$q_d(x, y) = x^2 - y^2, \quad r_d(x, y) = 2xy, \quad q_n(x, y) = x + 1, \quad r_n(x, y) = y$$

and using the definition of q and r in (6–7), we have:

$$q(x, y, k_d, k_n) = k_d(x^2 - y^2) + k_n(x + 1), \quad r(x, y, k_d, k_n) = 2k_dxy + k_ny.$$

Now we compute the Grobner basis for the ideal $\langle q, r \rangle$ using the graded reversed lexicographic order with $x > y > k_d > k_n$ and obtain $\{g_1, g_2, g_3, g_4\}$, where:

$$\begin{aligned} g_1(x, y, k_d, k_n) &= 2xyk_d + yk_n \quad (= r) \\ g_2(x, y, k_d, k_n) &= x^2k_d - y^2k_d + xk_n + k_n \quad (= q) \\ g_3(x, y, k_d, k_n) &= x^2yk_n + y^3k_n + 2xyk_n \\ g_4(x, y, k_d, k_n) &= 2y^3k_d - xyk_n - 2yk_n \end{aligned}$$

Now we homogenize of the polynomials g_i , using the procedure indicated in the Introduction:

$$g_1^h = z^3g_1(x/z, y/z, k_d/z, k_n/z) = 2xyk_d + yzk_n \quad (18)$$

$$g_2^h = z^3g_2(x/z, y/z, k_d/z, k_n/z) = x^2k_d - y^2k_d + xzk_n + z^2k_n \quad (19)$$

$$g_3^h = z^4g_3(x/z, y/z, k_d/z, k_n/z) = x^2yk_n + y^3k_n + 2xyzk_n \quad (20)$$

$$g_4^h = z^4g_4(x/z, y/z, k_d/z, k_n/z) = 2y^3k_d - xyk_n - 2yz^2k_n \quad (21)$$

The PjRL is the set of projective algebraic varieties \mathcal{W}_k generated by the four polynomials g_i^h presented in Eqs. (18–21) above, for each $k = k_n/k_d \in \mathbb{P}^1(\mathbb{R})$.

- Initial points of the PjRL (\mathcal{W}_0): Setting $k_n = 0$ and $k_d = 1$ in Eqs. (18–21), we obtain: $g_1^h = 2xy$, $g_2^h = x^2 - y^2$, $g_3^h = 0$ and $g_4^h = 2y^3$, and we see that the simultaneous solution for $g_i^h = 0$ is $(0, 0, z)$ where $z \in \mathbb{R}$. The homogeneous coordinate for this point is $(0 : 0 : z)$, but since $(0 : 0 : 0)$ is undefined we need $z \neq 0$ and the “unique” homogeneous coordinate possible is $(0 : 0 : 1)$. Then we have:

$$\mathcal{W}_0 = \{(0 : 0 : 1)\},$$

which represents the point $(0, 0)$ in the affine plane (XY) .

- Terminal points of the PjRL (\mathcal{W}_∞): Setting $k_n = 1$ and $k_d = 0$ in Eqs. (18–21), we obtain: $g_1^h = yz$, $g_2^h = xz + z^2$, $g_3^h = x^2y + y^3 + 2xyz$ and $g_4^h = -xyz - 2yz^2$. To solve $g_i^h = 0$, we can simplify the set of equations by recalculating a new Grobner basis with this set of polynomials. The new Grobner basis has three polynomials: $g_1^h = yz$, $g_2^h = xz + z^2$, $g_3^h = x^2y + y^3$. Then we have two kinds of points:
 1. Points at the affine plane XY ($z = 1$): In this case we get: $g_1 = y$, $g_2 = x + 1$, and $g_3 = y(x^2 + y^2)$. We see that the unique solution possible is $(-1, 0, 1)$ whose homogeneous coordinate is $(-1 : 0 : 1)$, and this (homogeneous) point corresponds to $(-1, 0)$ in the affine plane XY .
 2. Points at the infinite plane H ($z = 0$): In this case we obtain: $g_1 = 0$, $g_2 = 0$, and $g_3 = x^2y + y^3 = y(x^2 + y^2)$. We must have $y = 0$ and the unique possible nonzero solution is $(x, 0, 0)$, $x \neq 0$ whose homogeneous coordinate is $(x : 0 : 0) = (1 : 0 : 0)$, which corresponds to the intersection point of the *horizontal* lines in the plane XY . By the RL plot for this $G(s)$ (Figure 6) we see that this is the direction of the asymptotes when $k \rightarrow \pm\infty$.

We then have:

$$\mathcal{W}_\infty = \{(-1 : 0 : 1), (1 : 0 : 0)\}.$$

- Intermediary points of the PjRL (\mathcal{W}_λ): Setting $k_n = \lambda \neq 0$ and $k_d = 1$ in Eqs. (18–21) we obtain: $g_1^h = 2xy + yz\lambda$, $g_2^h = x^2 - y^2 + xz\lambda + z^2\lambda$, $g_3^h = \lambda(x^2y + y^3 + 2xyz)$, $g_4^h = 2y^3 - xyz\lambda - 2yz^2\lambda$. Computing a new Grobner basis with this set of polynomials we obtain three polynomials:

$$g_1^h = 2xy + yz\lambda, \quad g_2^h = x^2 - y^2 + xz\lambda + z^2\lambda, \quad g_3^h = x^2y + y^3 + 2xyz \quad (22)$$

We easily see that $z = 0$ will imply $x = y = 0$ and, since we are interested in nonzero solutions, we must have $z \neq 0$ (implying that doesn't exist intermediary points in the infinite plane H). To see the graph in the XY plane, we set $z = 1$ in these polynomials and obtain $g_1 = y(2x + \lambda)$, $g_2 = x^2 - y^2 + \lambda(x + 1)$, and $g_3 = y(x^2 + y^2 + 2x) = y[(x + 1)^2 + y^2 - 1]$. It is easy to see that solving $g_i = 0$ we obtain the same set of affine varieties \mathcal{V}_λ , $\lambda \in \mathbb{R}$, that represents the RL for $G(s)$ in the affine XY plane, as shown in Figure 6.

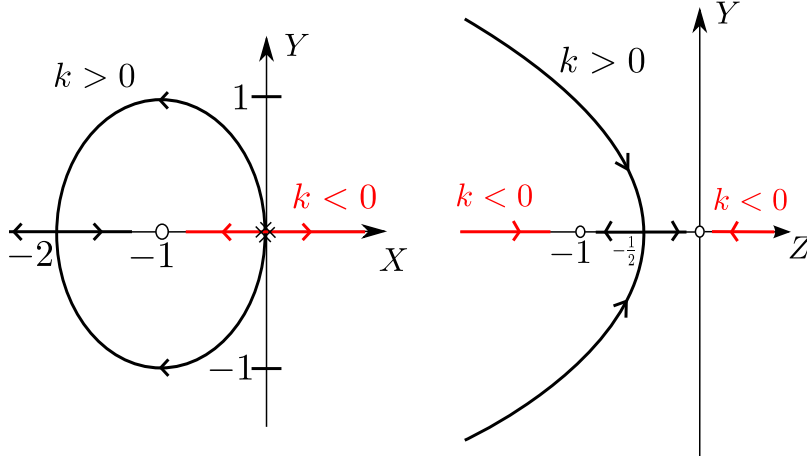


Figure 6: Conventional and complementary RL for $G(s) = (s + 1)/s^2$

We now analyze the question of obtaining the “complementary RL”, that corresponds to the intersection of the PjRL with ZY plane. We note that switching the roles of the x and z axis in the projective plane, the coordinate $(x : y : z)$ will become $(z : y : x)$. Then, re-analyzing the points calculated above, we have:

- Initial points: The projective point $\mathcal{W}_0 = \{(0 : 0 : 1)\}$ will become $\mathcal{W}'_0 = \{(1 : 0 : 0)\}$ and this corresponds to a point at infinity in the affine plane ZY (intersection of all horizontal lines).
- Terminal points: The projective points $\mathcal{W}_\infty = \{(-1 : 0 : 1), (1 : 0 : 0)\}$ will become $\mathcal{W}'_\infty = \{(1 : 0 : -1) = (-1 : 0 : 1), (0 : 0 : 1)\}$. This corresponds to the points $(-1, 0)$ and $(0, 0)$ in the affine plane ZY .
- Intermediary Points: To obtain the intermediary points \mathcal{W}'_λ in the affine plane ZY we set $x = 1$ in (22) to obtain:

$$\begin{aligned} g_1(y, z) &= 2y + yz\lambda = y(2 + z\lambda) \\ g_2(y, z) &= 1 - y^2 + z\lambda + z^2\lambda \\ g_3(y, z) &= y + y^3 + 2yz = y(1 + y^2 + 2z) \end{aligned}$$

We only have two cases ($y = 0$ and $y \neq 0$):

1. $y = 0$: In this case we are left only with $g_2(0, z) = 0$ or $\lambda z^2 + \lambda z + 1 = 0$ or $z^2 + z + 1/\lambda = 0$, whose roots are $z_{1,2} = (-1 \pm \sqrt{1 - 4/\lambda})/2$.
2. $y \neq 0$: In this case, from $g_3(y, z) = 0$ we have $y^2 + 2z + 1 = 0$, which is the parabola $z = -y^2/2 - 1/2$.

The plot for this set of equations in plane ZY is shown in Figure 6.

Example 3.4. Let be $G(s) = \frac{1}{s((s+4)^2 + 4^2)}$. In this case we have:

$$q = k_d(x^3 - 3xy^2 + 8x^2 - 8y^2 + 32x) + k_n, \quad r = k_d(-y^3 + 3yx^2 + 16xy + 32y)$$

and computing the Grobner basis using the graded reversed lexicographic order with $x > y > z > k_d > k_n$, we obtain the following set of homogenized Grobner polynomials:

$$g_1^h = 3x^2yk_n - y^3k_n + 16xyzk_n + 32yz^2k_n \quad (23)$$

$$g_2^h = 3x^2yk_d - y^3k_d + 16xyzk_d + 32yz^2k_d \quad (24)$$

$$g_3^h = x^3k_d - 3xy^2k_d + 8x^2zk_d - 8y^2zk_d + 32xz^2k_d + k_nz^3 \quad (25)$$

$$g_4^h = 24xy^3k_d + 64y^3zk_d - 64xyz^2k_d + 256yz^3k_d - 9yz^3k_n \quad (26)$$

$$g_5^h = 24y^5k_d - 320y^3z^2k_d + 1280xyz^3k_d - 27xyz^3k_n + 4096yz^4k_d - 72yz^4k_n \quad (27)$$

- Initial points (\mathcal{W}_0): In spite of knowing that the initial points are the roots of $d(s)$ we calculate them here just as a matter of checking the theory. Setting $k_d = 1$ and $k_n = 0$ in the polynomials above we get:

$$\begin{aligned}
g_1^h &= 0 \\
g_2^h &= 3x^2y - y^3 + 16xyz + 32yz^2 \\
g_3^h &= x^3 - 3xy^2 + 8x^2z - 8y^2z + 32xz^2 \\
g_4^h &= 24xy^3 + 64y^3z - 64xyz^2 + 256yz^3 \\
g_5^h &= 24y^5 - 320y^3z^2 + 1280xyz^3 + 4096yz^4
\end{aligned}$$

We note that doesn't exists initial points at infinity, since if we choose $z = 0$, by $g_5^h = 0$ we necessarily have $y = 0$ and, by $g_3^h = 0$ we get $x = 0$, but $(0 : 0 : 0)$ is not valid as solution. So, all the initial points are onto affine XY plane; then setting $z = 1$ in the homogeneous polynomials above we obtain the set of (non-homogeneous) polynomials:

$$\begin{aligned}
g_2 &= 3x^2y - y^3 + 16xy + 32y = y(3x^2 - y^2 + 16x + 32) \\
g_3 &= x^3 - 3xy^2 + 8x^2 - 8y^2 + 32x \\
g_4 &= 24xy^3 + 64y^3 - 64xy + 256y = y(24xy^2 + 64y^2 - 64x + 256) \\
g_5 &= 24y^5 - 320y^3 + 1280xy + 4096y = y(24y^4 - 320y^2 + 1280x + 4096)
\end{aligned}$$

We have two cases, namely $y = 0$ or $y \neq 0$; if we set $y = 0$ the only possible real value for x is obtained from $g_3 = 0$ and it is $x = 0$. So the first initial point is $(0 : 0 : 1)$. Considering $y \neq 0$ in the polynomials above we can rewrite them as:

$$\begin{aligned}
g_2/y &= 3x^2 - y^2 + 16x + 32 \\
g_3 &= x^3 - 3xy^2 + 8x^2 - 8y^2 + 32x \\
g_4/y &= 24xy^2 + 64y^2 - 64x + 256 \\
g_5/y &= 24y^4 - 320y^2 + 1280x + 4096
\end{aligned}$$

and if we compute a Grobner basis for the set of polynomials above we obtain:

$$\begin{aligned}
h_1 &= 3x^2 - y^2 + 16x + 32 \\
h_3 &= 3xy^2 + 8y^2 - 8x + 32 \\
h_4 &= 3y^4 - 40y^2 + 160x + 512
\end{aligned}$$

To solve the set of equations $h_1 = h_2 = h_3 = 0$, we can eliminate y^2 from h_1 and h_2 to obtain the equation $x^3 + 8x^2 + 24x + 32 = 0$ whose only real solution is $x = -4$, and this implies $y = \pm 4$. Finally we have the following set of initial points:

$$\mathcal{W}_0 = \{(0 : 0 : 1), (-4 : 4 : 1), (-4 : -4 : 1)\}$$

- Terminal points (\mathcal{W}_∞): Setting $k_d = 0$ and $k_n = 1$ in Eqs. (23–27) we obtain:

1. Points onto affine plane XY ($z = 1$): This will make $g_3^h = 1$, what means that doesn't exist terminal points onto affine plane XY .
2. Points at infinity ($z = 0$): We are left, from (23), only with $3x^2y - y^3 = y(3x^2 - y^2) = 0$. If $y = 0$, we will have the solution $(x : 0 : 0) = (1 : 0 : 0)$; if $y \neq 0$, we will have $y^2 = 3x^2$, that results in points $(1 : \pm\sqrt{3} : 0)$. So, the terminal points are:

$$\mathcal{W}_\infty = \{(1 : 0 : 0), (1 : \sqrt{3} : 0), (1 : -\sqrt{3} : 0)\}$$

- Intermediary points (\mathcal{W}_λ): Setting $k_d = 1$, $k_n = \lambda \neq 0$ in Eqs. (23–27), and computing a new Grobner basis we have the set below with only two polynomials:

$$g_1^h = 3x^2y - y^3 + 16xyz + 32yz^2, \quad \text{and} \quad g_2^h = z^3\lambda + x^3 - 3xy^2 + 8x^2z - 8y^2z + 32xz^2 \quad (28)$$

In this example we will only analyze the complementary RL (in plane ZY), and for that we will recalculate the initial and terminal points, just by switching the position of x and z in \mathcal{W}_0 and \mathcal{W}_∞ . So the new initial and terminal points are:

$$\mathcal{W}_0' = \{(1 : 0 : 0), (-1/4 : -1 : 1), (-1/4 : 1 : 1)\}$$

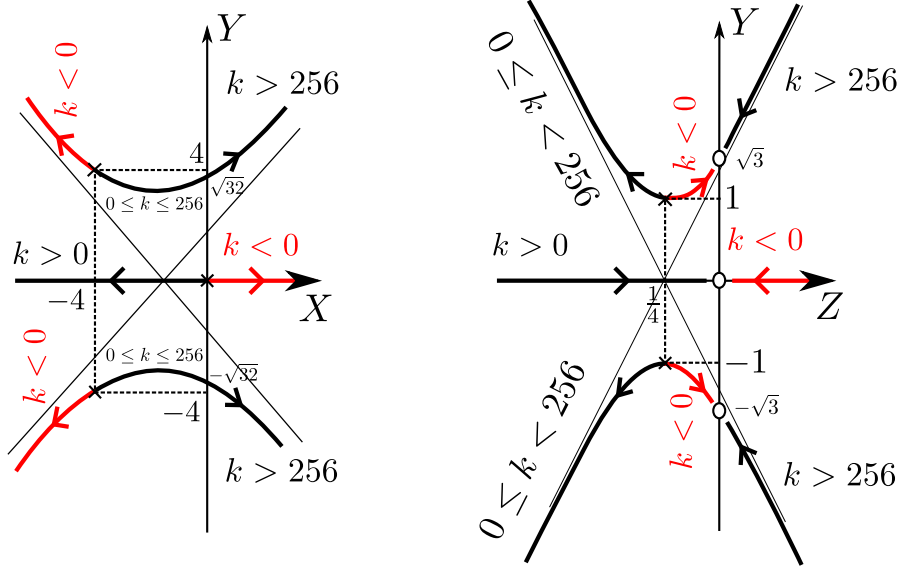


Figure 7: Conventional and complementary RL for $G(s) = \frac{1}{s((s+4)^2+4)}$

and

$$\mathcal{W}'_{\infty} = \{(0 : 0 : 1), (0 : \sqrt{3} : 1), (0 : -\sqrt{3} : 1)\}.$$

To evaluate the intermediary points we set $x = 1$ in the polynomials shown in (28) to obtain:

$$\begin{aligned} h_1 &= 3y - y^3 + 16yz + 32yz^2 = y(3 - y^2 + 16z + 32z^2) \\ h_2 &= z^3\lambda - 8y^2z - 3y^2 + 32z^2 + 8z + 1 \end{aligned}$$

And clearly we have two cases to consider, namely $y = 0$ and $y \neq 0$:

- $y = 0$ will imply $h_1 = 0$ and $h_2 = z^3\lambda + 32z^2 + 8z + 1$. We easily see that the cubic polynomial h_2 always have one real and two complex roots. The real root varies with λ as shown in Figure 7.
- $y \neq 0$ will imply $(-h_1/y) = y^2 - 32z^2 - 16z - 3$, so we have the polynomials:

$$\begin{aligned} -h_1/y &= y^2 - 32z^2 - 16z - 3 \\ h_2 &= z^3\lambda - 8y^2z - 3y^2 + 32z^2 + 8z + 1 \end{aligned}$$

Now if we compute again a Grobner basis for this set we will have:

$$\begin{aligned} l_1 &= y^2 - 32z^2 - 16z - 3 \\ l_2 &= (\lambda - 256)z^3 - 192z^2 - 64z - 8 \end{aligned}$$

We note that $l_1 = 0$ represents the hyperbole shown in Figure 7, while $l_2 = 0$ determines as z depends on λ . We note that when $\lambda \neq 256$, l_2 is a cubic polynomial with one real and two complex roots; moreover the real root explodes to infinity when $\lambda \rightarrow 256$. We also note that 256 is the value of the gain k where the RL crosses the Y axis (in plane XY).

Example 3.5. We now consider a case where the degree of d is equal the degree of n . The main point here is that the polynomial $d(s) + kn(s)$ may decrease its degree for some (finite) value of k . To simplify the PjRL plot we will consider a simple rational function G defined as:

$$G(s) = \frac{1 - s^2}{1 + s^2}$$

In this example we easily see that $(1 + s^2) + (1 - s^2) = 2$, so the degree of $d(s) + kn(s)$ is zero for $k = 1$, and we have no finite roots. To analyze the PjRL we evaluate the polynomials q and r , defined in (6-7), which in this case are:

$$q = k_d(1 + x^2 - y^2) + k_n(1 - x^2 + y^2), \quad \text{and} \quad r = k_d(2xy) + k_n(-2xy)$$

Computing the Grobner basis using the graded reversed lexicographic order with $x > y > z > k_d > k_n$, we obtain the following set of homogenized Grobner polynomials:

$$g_1^h = xyk_n \quad (29)$$

$$g_2^h = xyk_d \quad (30)$$

$$g_3^h = k_d(z^2 + x^2 - y^2) + k_n(z^2 + y^2 - x^2) \quad (31)$$

$$g_4^h = y(y^2k_d - y^2k_n - z^2k_d - z^2k_n) \quad (32)$$

- Initial points (\mathcal{W}_0): Using $k_d = 1$ and $k_n = 0$ in equations (29–32) above we obtain:

$$g_2^h = xy, \quad g_3^h = z^2 + x^2 - y^2, \quad \text{and} \quad g_4^h = y(y^2 - z^2)$$

and the unique (non-null) possible solution is $x = 0$, $y \neq 0$ and $y^2 = z^2$; so the homogeneous coordinates for \mathcal{W}_0 is:

$$\mathcal{W}_0 = \{(0 : 1 : 1), (0 : -1 : 1)\}$$

On the semi-sphere of radius one, we have:

$$\mathcal{W}_0 = \{(0, 1/\sqrt{2}, 1/\sqrt{2}), (0, -1/\sqrt{2}, 1/\sqrt{2})\}$$

- Terminal points (\mathcal{W}_∞): Using $k_d = 0$ and $k_n = 1$ in equations (29–32) above we obtain:

$$g_1^h = xy, \quad g_3^h = z^2 - x^2 + y^2, \quad \text{and} \quad g_4^h = -y(y^2 + z^2)$$

and the unique (non-null) possible solution is $x \neq 0$, $y = 0$ and $x^2 = z^2$; so the homogeneous coordinates for \mathcal{W}_∞ is:

$$\mathcal{W}_\infty = \{(1 : 0 : 1), (-1 : 0 : 1)\}$$

On the semi-sphere of radius one, we have:

$$\mathcal{W}_\infty = \{(1/\sqrt{2}, 0, 1/\sqrt{2}), (-1/\sqrt{2}, 0, 1/\sqrt{2})\}$$

- Intermediary points (\mathcal{W}_λ): Using $k_d = 1$ and $k_n = \lambda$ in equations (29–32) and recalculating the Grobner basis we get:

$$g_1^h = xy \quad (33)$$

$$g_2^h = (\lambda - 1)x^2 + (1 - \lambda)y^2 - (1 + \lambda)z^2 \quad (34)$$

$$g_3^h = y[-(1 - \lambda)y^2 + (\lambda + 1)z^2] \quad (35)$$

We have two cases to consider, namely, $x = 0, y \neq 0$ and $x \neq 0, y = 0$ (the case $x = 0$ and $y = 0$ will imply $z = 0$ which is not valid as a solution). Also, to plot the PjRL over the semi-sphere of radius one we have to consider the restriction $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Then we have:

1. $x = 0$, $y \neq 0$ and $y^2 + z^2 = 1$. Using equations (33–35) defined above, we necessarily have $(1 - \lambda)y^2 - (1 + \lambda)z^2 = 0$, and then we get:

$$y = \pm \sqrt{\frac{1 + \lambda}{2}}, \quad \text{and} \quad z = \sqrt{\frac{1 - \lambda}{2}}$$

So, considering $\lambda > 0$, we must have $0 < \lambda \leq 1$ and for $\lambda = 1$ we have the point at infinity $(0, 1, 0) = (0, -1, 0)$.

2. $x \neq 0$, $y = 0$ and $x^2 + z^2 = 1$. Using equations (33–35) defined above, we necessarily have $(\lambda - 1)x^2 - (1 + \lambda)z^2 = 0$, and then we get:

$$x = \pm \sqrt{\frac{\lambda - 1}{2\lambda}}, \quad \text{and} \quad z = \sqrt{\frac{\lambda - 1}{2\lambda}}$$

Again, considering $\lambda > 0$, we must have $\lambda \geq 1$, and for $\lambda = 1$ we get the point at infinity $(1, 0, 0) = (-1, 0, 0)$.

In Figure 8 we show the plot for the PjRL obtained in this example; as we can note, the PjRL has a discontinuity at infinity.

